

MULTILINEAR EMBEDDING – CONVOLUTION ESTIMATES ON SMOOTH SUBMANIFOLDS

WILLIAM BECKNER

ABSTRACT. Multilinear embedding estimates for the fractional Laplacian are obtained in terms of functionals defined over a hyperbolic surface. Convolution estimates used in the proof enlarge the classical framework of the convolution algebra for Riesz potentials to include the critical endpoint index, and provide new realizations for fractional integral inequalities that incorporate restriction to smooth submanifolds. Results developed here are modeled on the space-time estimate used by Klainerman and Machedon in their proof of uniqueness for the Gross-Pitaevskii hierarchy.

Analysis of the Gross-Pitaevskii hierarchy has led to the development and application of functional analytic mappings for the rigorous description of many-body interactions in quantum dynamics. Evolving from this framework is increased understanding for how Sobolev embedding and measures of fractional smoothness determine intrinsic size and growth estimates for functions and their Fourier transforms. Development of multilinear analysis increases understanding for genuinely n -dimensional aspects of Fourier analysis. In a formative and influential paper on uniqueness of solutions for the Gross-Pitaevskii hierarchy ([7]), Klainerman and Machedon prove a novel space-time estimate where the essential part of the proof corresponds to having uniform bounds for a three-dimensional convolution integral taken over a hyperbolic surface. Their result can be interpreted in the larger context of multilinear embedding where new end-point estimates are obtained. The resulting inequalities can be viewed as a step in the larger and dual program for understanding how smoothness controls restriction to a non-linear sub-variety (see [2])

The paradigm that underlies this objective to characterize control by multilinear embedding combines aspects of the Hardy-Littlewood-Sobolev inequality, the Hausdorff-Young inequality, Sobolev embedding and the uncertainty principle:

$$\int_{\mathbb{R}^n} |\hat{f}|^2 d\xi = \int_{\mathbb{R}^n} |f|^2 dx \leq C \left[\int_{\mathbb{R}^n} |(-\Delta/4\pi^2)^{\alpha/2} f|^p dx \right]^{2/p}$$

where $\alpha = n(1/p - 1/2) \geq 0$ and $1 < p \leq 2$. Our objective is to obtain multilinear embedding forms that extend this inequality and express capability for fractional smoothness to control restriction on a smooth submanifold. Results in this direction have already been given in [2], and initially reflect ideas of Calderón and Stein. The possibility that arises from the Klainerman-Machedon space-time estimate may be implicitly suggested by Stein's observation that "surface restriction" for the Fourier transform can sharpen estimates that use fractional integral arguments (see page 28 in [4]; pages 352–353, 374 in [10]): that is, surface integrals can be used as an auxiliary mechanism to characterize embedding action by Riesz potentials:

$$\int_S |x_1 + \cdots + x_m|^\sigma \left| \mathcal{F} \left[\prod |x_k|^{-\beta_k} * f \right] (x_1, \dots, x_m) \right|^r d\nu. \quad (1)$$

This is not directly "restriction phenomena" but rather a novel domain decomposition that results from adding new variables and submanifold restriction occurs for interior potential calculations used to define embedding forms. Still estimates of this kind have been used for

restriction-related arguments (see page 204 in [6]). The choice of a hyperbolic surface for this functional reflects both application-driven problems (free Schrödinger equation, Coulomb forces) and geometric invariance (conformal group, indefinite orthogonal group). The existence of distinguished directions for the surface will place limits on the range of multilinear embeddings that are considered and make the selection of uniform potentials and parameter constraints such as $m = n$ more intrinsic. This strategy will reinforce the underlying purpose for the paradigm: “*symmetry determines structure*”.

Outline of argument:

Consider m copies of \mathbb{R}^n and let f be in the Schwartz class $\mathcal{S}(\mathbb{R}^{mn})$. Define the Fourier transform

$$(\mathcal{F}f)(\xi) = \hat{f}(\xi) = \int e^{2\pi i \xi x} f(x) dx ,$$

and observe that on \mathbb{R}^n with $0 < \lambda < n$

$$\mathcal{F}: |x|^{-\lambda} \longrightarrow \pi^{-n/2+\lambda} \left[\frac{\Gamma(\frac{n-\lambda}{2})}{\Gamma(\frac{\lambda}{2})} \right] |x|^{-n+\lambda} .$$

For Δ_k = standard Laplacian on \mathbb{R}^n in the variable x_k , $\{\alpha\} = (\alpha_1, \alpha_2, \dots, \alpha_m)$, $0 < \alpha_k < n$, $\alpha = \sum \alpha_k$ for $k = 1$ to m , $1 < p$, $1 < p_* \leq 2$ and define

$$\begin{aligned} \Lambda_{p_*}(f; \{\alpha\}/p) &= \int_{\mathbb{R}^n \times \dots \times \mathbb{R}^n} \left| \prod_{k=1}^m (-\Delta_k/4\pi^2)^{\alpha_k/2p} f \right|^{p_*} dx_1 \dots dx_m \\ &= \int_{\mathbb{R}^n \times \dots \times \mathbb{R}^n} \left| \int e^{-2\pi i x \xi} \prod_{k=1}^m |\xi_k|^{\alpha_k/p} \hat{f} d\xi_1 \dots d\xi_m \right|^{p_*} dx_1 \dots dx_m . \end{aligned} \quad (2)$$

For $w \in \mathbb{R}^n$, $\tau > 0$ define

$$d\nu = \delta\left(\tau + \sum' |x_k|^2 - |x_m|^2\right) \delta\left(w - \sum x_k\right) dx_1 \dots dx_m$$

(here the prime on the symbol for sum, product or sequence indicates that the last term should be dropped); then for $\sigma > 0$, $r \geq 1$, $1 < p < \infty$, $1/p + 1/q = 1$, $r q \geq 2$, $1/p_* + 1/(r q) = 1$ and $\beta_k = n - \alpha_k/(pr)$

$$\begin{aligned} A \int & \left[\int |x_1 + \dots + x_m|^{\sigma/p} \left| \mathcal{F} \left[\prod |x_k|^{-\beta_k} * f \right] (x_1, \dots, x_m) \right|^r d\nu \right]^q dw d\tau \\ &= \int \left[\int |x_1 + \dots + x_m|^{\sigma/p} \prod |x_k|^{-\alpha_k/p} |\hat{f}|^r d\nu \right]^q dw d\tau \\ &\leq \sup_{w, \tau} \left[\int |x_1 + \dots + x_m|^\sigma \prod |x_k|^{-\alpha_k} d\nu \right]^{q/p} \int \left[\int |\hat{f}|^{r q} d\nu \right] dw d\tau \\ &\leq C \int |\hat{f}|^{r q} dx \leq C \left[\int |f|^{p_*} dx \right]^{r q/p_*} . \end{aligned} \quad (3)$$

This result can be rephrased in terms of multilinear embedding:

$$\left[\int \left[\int |x_1 + \dots + x_m|^{\sigma/p} |\hat{f}|^r d\nu \right]^q dw d\tau \right]^{p_*/(r q)} \leq \Lambda_{p_*}(f; \{\alpha\}/(pr)) . \quad (4)$$

Observe that for $p_* = 2$ and $pr \geq 2$ (equivalently $p \geq 2$), the range of values for $\alpha = \sum \alpha_k$ can be lowered from the limit used in [2]. The essential step for this argument is to determine the range of parameter values for which

$$\sup_{w, \tau} \int |x_1 + \dots + x_m|^\sigma \prod |x|^{-\alpha_k} dv \quad (5)$$

will be bounded.

Such results will extend the classical convolution for Riesz potentials

$$\int_S \frac{1}{|w - y|^\lambda} \frac{1}{|y|^\mu} dv$$

where S is a smooth submanifold in \mathbb{R}^n , $w \in \mathbb{R}^d$ and the objective is to bound the size of the integral by an inverse power of $|w|$ under suitable conditions on λ and μ . And in turn, these bounded convolution forms can serve as kernels to define new Stein-Weiss fractional integrals that characterize control by smoothness. Formula (5) is equivalent to

$$\sup_{w, \tau} |w|^\sigma \int \delta\left(\tau + \sum' |x_k|^2 - |x_m|^2\right) \delta\left(w - \sum x_k\right) \prod |x_k|^{-\alpha_k} dx_1 \dots dx_m.$$

In the case where $\sigma = 2 + \alpha - n(m - 1)$, this form has an invariance which makes it a function of only one variable:

$$\sup_w |w|^\sigma \int \delta\left(1 + \sum' |x_k|^2 - |x_m|^2\right) \delta\left(w - \sum x_k\right) \prod |x_k|^{-\alpha_k} dx_1 \dots dx_m. \quad (6)$$

Now for the case $\alpha = n(m - 1)$, this form becomes an extension of the classical convolution form

$$(g * f_1 * \dots * f_m)(w), \quad g \in L^1(\mathbb{R}^n), \quad f_k \in L^{n/\alpha_k}(\mathbb{R}^n)$$

which is uniformly continuous and in the class $C_o(\mathbb{R}^n)$ by using the Riemann-Lebesgue lemma for convolution. Here the convolution for Lebesgue classes is replaced by Riesz potentials, but the multivariable integration is constrained to be on a hyperbolic surface invariant under the action of the indefinite orthogonal group.

Effectively one obtains a new embedding kernel from formula (6)

$$\int \delta\left(1 + \sum' |x_k|^2 - |x_m|^2\right) \delta\left(w - \sum x_k\right) \prod |x_k|^{-\alpha_k} dx_1 \dots dx_m \leq \frac{C}{|w|^2} \quad (7)$$

which defines a map from $L^{n/(n-1)}(\mathbb{R}^n)$ to $L^n(\mathbb{R}^n)$. Any type of efficient proof for estimates of this type will require a reduction argument for the number of convolutions which in turn will place constraints on the range of possible values of parameters. Though at first glance seemingly specialized, the case of uniform potentials with $m = n$ captures essential features of this problem, including the 3-dimensional model calculation by Klainerman and Machedon, with small allowance for variation. In part, this dominant feature reflects the conversion of the required multivariable estimate to basically an n -dimensional calculation in terms of vector lengths:

$$\prod |x_k|^{-(n-1)} dx_1 \dots dx_n = d|x_1| \dots d|x_n| d\sigma$$

where $d\sigma$ denotes surface measure on n copies of S^{n-1} . Moreover, such a choice allows a natural reduction to low dimension when the number of convolutions is at least three. The strategy for developing this argument rests on three cornerstones: 1) reduction of number of convolutions in the estimate; 2) direct integration methods with focus on utilization of polar angle integration to obtain sharper dependence on the “free variable” $|w|$; 3) explicit calculations in two dimensions. Notice that the estimates (6) for the surface integrals do not depend on the embedding indices for Lebesgue classes.

Theorem 1. *Let $0 < \alpha_k < n$, $\alpha = \sum \alpha_k$, $m \geq 3$, $n \geq 3$ and $\rho = 2 + \alpha - (m-1)n$ for $0 < \rho < n$. For $m > \ell \geq 2$ assume that there are ℓ choices of the α_k 's having the property that $\ell n - 2 > \beta_{\ell,m} > (\ell-1)n - 2$ with $\beta_{\ell,m} = \sum \alpha_k$ for $k = m - \ell + 1$ to m . Then for $\sigma = 2 + \beta_{\ell,m} - (\ell-1)n$*

$$\begin{aligned} & \sup_w |w|^\rho \int \delta\left(1 + \sum' |x_k|^2 - |x_m|^2\right) \delta\left(w - \sum x_k\right) \prod |x_k|^{-\alpha_k} dx_1 \dots dx_m \\ & \leq C \sup_w |w|^\sigma \int \delta\left(1 + \sum' |x_k|^2 - |x_m|^2\right) \delta\left(w - \sum x_k\right) \prod |x_k|^{-\alpha_k} dx_{m-\ell+1} \dots dx_m \end{aligned} \quad (8)$$

where the sums and products in the second term are taken over the indices $k = m - \ell + 1$ to m .

Proof.

$$\begin{aligned} & |w|^\rho \int \delta\left(1 + \sum' |x_k|^2 - |x_m|^2\right) \delta\left(w - \sum x_k\right) \prod |x_k|^{-\alpha_k} dx_1 \dots dx_m \\ & = |w|^\rho \int \prod_1^{m-\ell} |x_k|^{-\alpha_k} \left|w - \sum_1^{m-\ell} x_k\right|^{-\sigma} \left[\left|w - \sum_1^{m-\ell} x_k\right|^\sigma \int \delta\left(1 + \sum' |x_k|^2 - |x_m|^2\right) \delta\left(w - \sum x_k\right) \times \right. \\ & \quad \left. \prod_{m-\ell+1}^m |x_k|^{-\alpha_k} dx_{m-\ell+1} \dots dx_m \right] dx_1 \dots dx_{m-\ell} \\ & \leq |w|^\rho \int \prod_1^{m-\ell} |x_k|^{-\alpha_k} \left|w - \sum_1^{m-\ell} x_k\right|^{-\sigma} dx_1 \dots dx_{m-\ell} \times \\ & \quad \sup_{w,\tau} \left[|w|^\sigma \int \delta\left(\tau + \sum'_{m-\ell+1} |x_k|^2 - |x_m|^2\right) \delta\left(w - \sum_{m-\ell+1} x_k\right) \prod_{m-\ell+1} |x_k|^{-\alpha_k} dx_{m-\ell+1} \dots dx_m \right] \\ & = C \sup_{w,\tau} \left[|w|^\sigma \int \delta\left(\sum'_{m-\ell+1} |x_k|^2 - |x_k|^2\right) \delta\left(w - \sum_{m-\ell+1} x_k\right) \prod_{m-\ell+1} |x_k|^{-\alpha_k} dx_{m-\ell+1} \dots dx_m \right] \end{aligned}$$

where

$$C = |w|^\rho \int \prod_1^{m-\ell} |x_k|^{-\alpha_k} \left|w - \sum_1^{m-\ell} x_k\right|^{-\sigma} dx_1 \dots dx_{m-\ell}$$

and can be calculated from the action of the Fourier transform on Riesz potentials. For the latter “sup” term, the new parameters w, τ were obtained by setting $\tau = 1 + \sum |x_k|^2$ for $1 \leq k \leq m - \ell$ and $w - \sum x_k$ ($1 \leq k \leq m - \ell$) $\rightsquigarrow w$. It must be recognized that the conditions given in the statement of the theorem may not be sufficient to guarantee that the “sup” over ℓ convolutions is finite. \square

The concern here will not be to treat the full range of parameters but to especially cover the cases $\ell = 2$ and 3 , and to establish estimates for uniform potentials with $\alpha_k = n - 1$ for $3 \leq m \leq n + 1$.

Theorem 2. *For $n \geq 2$, $\sigma = \alpha + \lambda + 2 - n$, $2(n-1) > \alpha + \lambda > n-1$ and $0 < \alpha < n-1$, $\lambda > 0$*

$$\Theta_n(w) = |w|^\sigma \int_{\mathbb{R}^n \times \mathbb{R}^n} \delta[1 + |x|^2 - |y|^2] \delta(w - x - y) |x|^{-\alpha} |y|^{-\lambda} dx dy \quad (9)$$

is uniformly bounded for $w \in \mathbb{R}^n$.

Proof. Observe that since $|y| \geq 1$, there is no upper bound for λ in this computation; but this calculation may be used for embedding where $0 < \sigma < n$ so there can be an effective upper bound resulting from application.

$$\begin{aligned}
\Theta_{n,\alpha}(w) &= |w|^\sigma \int_{\mathbb{R}^n} \delta[1 + |w - y|^2 - |y|^2] |w - y|^{-\alpha} |y|^{-\lambda} dy \\
&= \frac{2\pi^{(n-1)/2}}{\Gamma((n-1)/2)} |w|^\sigma \int_0^1 \int_1^\infty \delta(1 + |w|^2 - 2|w|ru) (r^2 - 1)^{-\alpha/2} r^{n-\lambda-1} (1 - u^2)^{(n-3)/2} dr du \\
&= \frac{2^{\alpha+\lambda-n} \pi^{(n-1)/2}}{\Gamma((n-1)/2)} \left[\frac{|w|^2}{1 + |w|^2} \right]^{\alpha+\lambda-n+1} \int_0^1 u^{(\alpha+\lambda-n+1)/2-1} (1 - u)^{(n-3)/2} (1 - \beta(w)u)^{-\alpha/2} du \\
&= C_n \left[\frac{|w|^2}{1 + |w|^2} \right]^{\alpha+\lambda-n+1} F(\alpha/2, (\alpha + \lambda - n + 1)/2; (\alpha + \lambda)/2; \beta(w))
\end{aligned} \tag{10}$$

where F denotes the hypergeometric function, $\beta(w) = \frac{4|w|^2}{(1+|w|^2)^2} \leq 1$ and

$$C_n = 2^{\alpha+\lambda-n} \pi^{(n-1)/2} \frac{\Gamma((\alpha + \lambda - n + 1)/2)}{\Gamma((\alpha + \lambda)/2)}.$$

Then

$$\theta_n(w) \leq 2^{\alpha+\lambda-n} \pi^{(n-1)/2} \frac{\Gamma((\alpha + \lambda - n + 1)/2) \Gamma((n-1-\alpha)/2)}{\Gamma((n-1)/2) \Gamma(\lambda/2)} \left[\frac{|w|^2}{1 + |w|^2} \right]^{\alpha+\lambda-n+1}. \tag{11}$$

Hence $\Theta_n(w)$ is bounded for $\lambda > 0$, $0 < \alpha < n - 1$ and $\alpha + \lambda > n - 1$. \square

This result demonstrates that at least three convolutions are needed to account for uniform potentials with $\alpha_k = n - 1$. In the case of dimension two, one can explicitly compute that

$$\begin{aligned}
&|w|^2 \int_{\mathbb{R}^n \times \mathbb{R}^2} \delta(1 + |x|^2 - |y|^2) \delta(w - x - y) |x|^{-1} |y|^{-1} dx dy \\
&= \frac{|w|^2}{1 + |w|^2} \int_0^1 u^{-1/2} (1 - u)^{-1/2} (1 - \beta(w)u)^{-1/2} du \\
&= \frac{2|w|^2}{1 + |w|^2} K(\sqrt{\beta(w)}) \simeq -\ln \sqrt{1 - \beta(w)} \quad \text{for } \beta(w) \simeq 1
\end{aligned}$$

where K denotes the complete elliptic integral.

Theorem 3. For $n \geq 2$, $\sigma = 2 + \alpha_1 + \alpha_2 + \lambda - 2n$ and $3n - 2 > \alpha_1 + \alpha_2 + \lambda > 2n - 2$, $\alpha_1 + \lambda > n - 1$, $\alpha_2 + \lambda > n - 1$, $2n - 1 > \alpha_1 + \alpha_2$

$$\Delta_n(w) = |w|^\sigma \int_{\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n} \delta[1 + |w|^2 + |z|^2 - |y|^2] \delta(w - x - y - z) |x|^{-\alpha_1} |z|^{-\alpha_2} |y|^{-\lambda} dx dy dz \tag{12}$$

is uniformly bounded for $w \in \mathbb{R}^n$.

Proof. Using translations and applying the two delta functions

$$\begin{aligned}
\Delta_n(w) &= |w|^\sigma \int_{\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n} \delta[1 + |w|^2 + |z|^2 - |y|^2] \delta(x + y + z) |x + w|^{-\alpha_1} |z|^{-\alpha_2} |y|^{-\lambda} dx dy dz \\
&= |w|^\sigma \int_{\mathbb{R}^n \times \mathbb{R}^n} \delta[1 + |x - w|^2 + |x - y|^2 - |y|^2] |x - w|^{-\alpha_1} |x - y|^{-\alpha_2} dx dy dz \\
&= |w|^\sigma \int_{\mathbb{R}^n \times \mathbb{R}^n} \delta[1 + |x - w|^2 + |x|^2 - 2x \cdot y] |x - w|^{-\alpha_1} |y|^2 - 1 - |x - w|^2|^{-\alpha_2/2} |y|^{-\lambda} dx dy \\
&= s_n |w|^\sigma \int_{\mathbb{R}^n} \int_0^1 \int_1^\infty \delta[1 + |x - w|^2 + |x|^2 - 2|x|ur] |x - w|^{-\alpha_1} |r^2 - 1 - |x - w|^2|^{-\alpha_2/2} \times \\
&\quad r^{n-\lambda-1} (1 - u^2)^{(n-3)/2} dr du dx \\
&= s_n |w|^\sigma \int_{\mathbb{R}^n} \int_0^1 (2|x|u)^{\lambda-n+\alpha_2} [1 + |x - w|^2 + |x|^2]^{n-\lambda-1-\alpha_2} |x - w|^{-\alpha_1} \times \\
&\quad \left[1 - \frac{4|x|^2(1 + |x - w|^2)u^2}{(1 + |x - w|^2 + |x|^2)^2} \right]^{-\alpha_2/2} (1 - u^2)^{(n-3)/2} du dx \\
&= 2^{\lambda-n+\alpha_2-1} s_n |w|^\sigma \int_{\mathbb{R}^n} |x|^{\lambda+\alpha_2-n} |x - w|^{-\alpha_1} [1 + |x - w|^2 + |x|^2]^{n-1-\lambda-\alpha_2} \times \\
&\quad \left[\int_0^1 u^{(\lambda+\alpha_2-n+1)/2-1} (1 - u)^{(n-3)/2} (1 - \beta(x, w)u)^{-\alpha_2/2} du \right] dx
\end{aligned}$$

where $s_n = 2\pi^{(n-1)/2} / \Gamma((n-1)/2)$

$$\beta(x, w) = \frac{4|x|^2(1 + |w - x|^2)}{(1 + |w - x|^2 + |x|^2)^2}, \quad 1 - \beta(x, w) = \left[\frac{1 + |w - x|^2 - |x|^2}{1 + |w - x|^2 + |x|^2} \right]^2.$$

Then

$$\begin{aligned}
\Delta_n(w) &= 2^{\lambda+\alpha_2-n-1} s_n \frac{\Gamma((\lambda + \alpha_2 - n + 1)/2)}{\Gamma((\lambda + \alpha_2)/2)} |w|^\sigma \times \\
&\quad \int_{\mathbb{R}^n} |x|^{\lambda+\alpha_2-n} |x - w|^{-\alpha_1} [1 + |x - w|^2 + |x|^2]^{n-1-\lambda-\alpha_2} \times \\
&\quad {}_2F_1\left(\alpha_2/2, (\lambda + \alpha_2 - n + 1)/2; (\lambda + \alpha_2)/2; \beta(x, w)\right) dx.
\end{aligned}$$

Step 1: The original calculation here is symmetric in α_1 and α_2 . Suppose that for one of these values, say α_2 , that $\lambda + \alpha_2 > n - 1$ and $\alpha_2 < n - 1$. The integral in u above is bounded by

$$\frac{\Gamma((\lambda + \alpha_2 - n + 1)/2) \Gamma((n - 1 - \alpha_2)/2)}{\Gamma(\lambda/2)};$$

then to see that Δ_2 is uniformly bounded in w , it suffices to show that

$$|w|^\sigma \int_{\mathbb{R}^n} |x|^{\lambda-n+\alpha_2} |x - w|^{-\alpha_1} [1 + |x - w|^2 + |x|^2]^{n-\lambda-1-\alpha_2} dx$$

is bounded. This term is less than

$$|w|^\sigma \int_{\mathbb{R}^n} |x - w|^{-\alpha_1} [1 + |x - w|^2 + |x|^2]^{-(\lambda + \alpha_2 - n)/2 - 1} dx$$

which is bounded since the sum of the “powers of $|x|$ ” is less than $2n$ (by hypothesis)

$$\alpha_1 + \alpha_2 + \lambda - n + 2 < 2n$$

so that

$$\begin{aligned} |w|^\sigma \int_{\mathbb{R}^n} |x - w|^{-\alpha_1} [1 + |x - w|^2 + |x|^2]^{-(\lambda + \alpha_2 - n)/2 - 1} dx \\ \leq |w|^\sigma \int_{\mathbb{R}^n} |x - w|^{-\gamma_1} |x|^{-\gamma_2} dx = C \end{aligned}$$

where $0 < \gamma_1, \gamma_2 < n$ and $\gamma_1 + \gamma_2 = \alpha_1 + \alpha_2 + \lambda - n + 2$.

Step 2: Assume that $\alpha_2 \geq n - 1$ (thus insuring $\lambda + \alpha_2 > n - 1$), $\alpha_1 + \lambda > n - 1$, and $2n - 1 > \alpha_1 + \alpha_2$; choose γ so that

$$\min \left\{ n - \alpha_1, \alpha_1 + \alpha_2 + \lambda - 2(n - 1), 1 \right\} > 2\gamma > \max \left\{ 1 - \alpha_2, \alpha_2 - (n - 1) \right\} \geq 0 ;$$

apply the estimate

$$(1 - \beta(x, w)u)^{-\alpha_2/2} \leq (1 - \beta(x, w))^{-\gamma} (1 - u)^{-(\alpha_2 - 2\gamma)/2}$$

and the integral in u becomes

$$\begin{aligned} \int_0^1 u^{(\lambda + \alpha_2 - n + 1)/2 - 1} (1 - u)^{(n - 1 - \alpha_2 + 2\gamma)/2 - 1} du \\ = \frac{\Gamma((\lambda + \alpha_2 - n + 1)/2) \Gamma((n - 1 + 2\gamma - \alpha_2)/2)}{\Gamma((\lambda + 2\gamma)/2)} . \end{aligned}$$

For $\Delta_n(w)$ to be bounded, the form

$$|w|^\sigma \int_{\mathbb{R}^n} |x|^{\lambda - n + \alpha_2} |x - w|^{-\alpha_1} [1 + |x - w|^2 + |x|^2]^{n - \lambda - 1 - \alpha_2 + 2\gamma} |1 + |x - w|^2 - |x|^2|^{-2\gamma} dx$$

must be bounded, and this term is less than

$$|w|^\sigma \int_{\mathbb{R}^n} |x - w|^{-\alpha_1} [1 + |x - w|^2 + |x|^2]^{-(\lambda + \alpha_2 - n)/2 + 2\gamma - 1} |1 + |x - w|^2 - |x|^2|^{-2\gamma} dx .$$

Choose the x_1 direction to be along that for w_1 and rearrange in the variable x_1 using for $x = (x_1, x') \in \mathbb{R}^n$

$$\int_{\mathbb{R}^n} f(x_1, x') g(x_1, x') h(x_1, x') dx_1 dx' \leq \int_{\mathbb{R}^n} f_{\#}(x_1, x') g_{\#}(x_1, x') h_{\#}(x_1, x') dx_1 dx'$$

where $f_{\#}(x_1, x')$ is the equimeasurable decreasing rearrangement of $|f(x_1, x')|$ in the variable $x_1 \in \mathbb{R}$. After applying this rearrangement argument and making a change of variables to

remove the dependence on $|w|$, the integral above is less than

$$\begin{aligned}
& |w|^{\sigma-2\gamma} \int_{\mathbb{R}^n} |x|^{-\alpha_1} \left[1 + 2|x|^2 + |w|^2/2 \right]^{-(\lambda+\alpha_2-n)/2+2\gamma-1} |x_1|^{-2\gamma} dx \\
& < \int_{\mathbb{R}^n} |x|^{-\alpha_1} \left[2|x|^2 + 1/2 \right]^{-(\lambda+\alpha_2-n)/2+2\gamma-1} |x_1|^{-2\gamma} dx \\
& = \int_{\mathbb{R}^n} \left(1 + |x_1|^2 \right)^{-\alpha_1/2} |x_1|^{-2\gamma} |x'|^{1-\alpha_1-2\gamma} \left[2|x'|^2 \left(1 + |x_1|^2 \right) + 1/2 \right]^{-(\lambda+\alpha_2-n)/2+2\gamma-1} dx \\
& = \int_{\mathbb{R}} \left(1 + |x_1|^2 \right)^{-(n-2\gamma)/2} |x_1|^{-2\gamma} dx_1 \int_{\mathbb{R}^{n-1}} |x'|^{1-\alpha_1-2\gamma} \left[2|x'|^2 + 1/2 \right]^{-(\lambda+\alpha_2-n)/2+2\gamma-1} dx'
\end{aligned}$$

For the given assumption on 2γ , these integrals are bounded thus insuring that $\Delta_n(w)$ is uniformly bounded and completes the proof of Theorem 3. \square

Taken together Theorems 1 and 3 include both a natural extension and an independent proof of the Klainerman-Machedon three-dimensional estimate:

$$|w|^2 \int_{\mathbb{R}^n \times \dots \times \mathbb{R}^n} \delta \left[\tau + \sum' |x_k|^2 - |x_n|^2 \right] \delta \left(w - \sum x_k \right) \prod |x_k|^{-(n-1)} dx_1 \dots dx_n \quad (13)$$

$$|w|^{n-1} \int_{\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n} \delta \left[\tau + |z|^2 + |x|^2 - |y|^2 \right] \delta(w - x - y - z) [|z| |x| |y|]^{-(n-1)} dx dy dz \quad (14)$$

are uniformly bounded in terms of the variable $\tau > 0$ and $w \in \mathbb{R}^n$ with $n > 1$.

These results form the basis for stating the principal multilinear embedding theorem. But it seems better to formulate this result in less than full generality since roughly the average value of α_k should be on the order of n . Hence the embedding result will be stated for the case of uniform potentials $\alpha_k = n - 1$ which satisfies all the required conditions for Theorems 1 and 3.

Theorem 4. *Let $\alpha_k = n - 1$ for $k = 1$ to m , $3 \leq m \leq n + 1$, $n \geq 2$, $\sigma = 2 + n - m$, $r \geq 1$, $1 < p < \infty$, $1/p + 1/q = 1$, $rq \geq 2$ and $1/p_* + 1/(rq) = 1$; then*

$$\begin{aligned}
& \left[\int_{\mathbb{R}^n \times \mathbb{R}_+} \left[\int |x_1 + \dots + x_m|^{\sigma/p} |\hat{f}|^r d\nu \right]^q dw d\tau \right]^{p_*/(rq)} \\
& \leq C \Lambda_{p_*} \left(f; \{\alpha\}/(pr) \right)
\end{aligned} \quad (15)$$

Proof. Apply Theorems 1 and 3 in the initial “Outline of Argument.”

Letting indices and parameters be given as above in Theorem 4, then duality gives for $g \in L^p(\mathbb{R}^n \times \mathbb{R}_+)$, $1 < p < \infty$:

Corollary. *For $D = \{x_k \in \mathbb{R}^n, k = 1 \text{ to } m : |x_m|^2 \geq \sum' |x_k|^2\}$*

$$\begin{aligned}
& \left[\int_D \left| g \left(\sum x_k, |x_m|^2 - \sum' |x_k|^2 \right) \right|^p \left| \sum x_k \right|^\sigma \prod |x_k|^{-(n-1)} dx_1 \dots dx_m \right]^{1/p} \\
& \leq C \left[\int_{\mathbb{R}^n \times \mathbb{R}_+} |g(w, \tau)|^p dw d\tau \right]^{1/p}
\end{aligned} \quad (16)$$

Following on the more limited choices of values for the $\{\alpha_k\}$'s in Theorem 4, it's possible to give an alternative proof with these values, e.g. $\alpha_k = n - 1$, for Theorem 3 using dimension reduction to show

$$\sup \Delta_n(w) \leq C \sup \Delta_2(w) .$$

The proof that $\Delta_2(w)$ is bounded comes easily from the latter part of the argument for Theorem 3.

By recasting the potential calculation given in the Appendix for [7] from a spherical surface to a hyperbolic surface, there is an implicit suggestion that the natural object of study for convolution forms given over surfaces may correspond to the type:

$$\sup_{w, \tau} |w|^p \int \delta\left(\tau + \sum'_{k_j} |x_k|^2 - |x_m|^2\right) \delta\left(w - \sum_{k'_j} x_k\right) \prod |x_k|^{-\alpha_k} dx_1 \dots dx_m \quad (17)$$

where $\{k_j\}$ and $\{k'_j\}$ may be different sequences. Relevance of specific forms will likely reflect application.

Observe from equation 2 that for $p_* = 2$, then

$$\Lambda_{p_*}(f; \{\alpha\}/pr) = \int_{\mathbb{R}^n \times \dots \times \mathbb{R}^n} \prod |\xi_k|^{2\alpha_k/(pr)} |\hat{f}(\xi)|^2 d\xi_1 \dots d\xi_m$$

and since $p_* = 2 = qr$, then $q \leq 2$ and $p \geq 2$ with $pr > 2$ for $r > 1$. In this case and using the hypothesis of Theorem 4, one has

$$\frac{2}{pr} \sum \alpha_k = \frac{2}{pr} m(n-1) < (m-1)n \text{ for } m = r, n+1 .$$

Hence Theorem 4 gives a multilinear embedding estimate where the embedding indices can fall below the critical lower bound expressed in the restriction results obtained in [2]. But still the standard Sobolev embedding estimates hold for iterated potentials so in that context it is also possible to move below the critical index for restriction. For $\alpha_k = \alpha_0$ for all k , $0 < \alpha_0 < n$, and $p = 2n/(n - \alpha_0)$

$$\left[\|f\|_{L^p(\mathbb{R}^{mn})} \right]^2 \leq c \Lambda_2\{\alpha\} < c \int_{\mathbb{R}^{mn}} \left| (-\Delta/4\pi^2)^{m\alpha_0/4} f \right|^2 dx$$

Implicit in the formulation of the problems treated here is the continuing development of new forms that characterize control by smoothness for size. As an example, and a consequence of the principal estimate obtained here, bounds for new Stein-Weiss integrals with a kernel determined by restriction to a smooth submanifold can be shown.

Starting from the convolution form used in Theorem 4

$$\int \delta\left(1 + \sum' |x_k|^2 - |x_m|^2\right) \delta\left(w - \sum x_k\right) \prod |x_k|^{-(n-1)} dx_1 \dots dx_m \leq \frac{C}{|x|^\sigma}$$

for $\sigma = 2 + n - m$, $3 \leq m \leq n + 1$, and $n \geq 2$, one obtains forms of Stein-Weiss type.

Theorem 5. Define for $\tau > 0$, and $w, v \in \mathbb{R}^n$

$$K_\tau(w, v) = \int_{\mathbb{R}^n \times \dots \times \mathbb{R}^n} \prod |x_k|^{-(n-1)} \left[\left| w - \sum x_k \right| \left| v - \sum x_k \right| \right]^{-(n+m)/2 + 1} \times \\ \delta\left[\tau + \sum' |x_k|^2 - |x_m|^2\right] dx_1 \dots dx_m ,$$

then for non-negative $f \in L^2(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} f(w) K_\tau(w, v) f(v) dw dv \leq c \int_{\mathbb{R}^n} |f|^2 dx \quad (18)$$

Proof. Apply Pitt's inequality and the uniform bounds obtained from Theorems 1 and 3. Here c is a generic constant. \square

In contrast to regular Stein-Weiss integrals, this kernel K_τ is not homogeneous with respect to dilation. By taking $\tau = 0$, the resulting kernel is homogeneous and satisfies the conditions given for the Stein-Weiss lemma from the Appendix in [1]. This allows in principle a formula for the optimal constant.

Corollary 1. *For the integral operator*

$$(Tf)(w) = \int_{\mathbb{R}^n} K_0(w, v) f(v) dv$$

that maps $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$

$$\|Tf\|_{L^2(\mathbb{R}^n)} \leq A\|f\|_{L^2(\mathbb{R}^n)}$$

with the optimal constant given by

$$A = \int_{\mathbb{R}^n} K_0(w, \hat{e}_1) |w|^{-n/2} dw \quad (19)$$

where \hat{e}_1 is a unit vector in the first coordinate direction.

Further extensions come from taking a non-negative integrable function on \mathbb{R}_+ and integrating out the surface delta function.

Corollary 2. *For $\varphi \geq 0$ with*

$$K_\varphi(w, v) = \int_{\mathbb{R}^n \times \dots \times \mathbb{R}^n} \prod |x_k|^{-(n-1)} \left[|w - \sum x_k| |v - \sum x_k| \right]^{-(n+1)/2 + 1} \varphi \left[|x_m|^2 - \sum' |x_k|^2 \right] dx_1 \dots dx_m$$

and $\int_0^\infty \varphi(t) dt = 1$, then

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} f(w) K_\varphi(w, v) f(v) dw dv \leq c \int_{\mathbb{R}^n} |f|^2 dx. \quad (20)$$

Overall these results are examples of the following general embedding estimate which gives “size control” at the L^2 level:

$$\int_{S \subset \mathbb{R}^{mn}} \left| f \left(\sum x_k \right) \right|^2 \prod |x_k|^{-\alpha_k} d\nu \leq c \int_{\mathbb{R}^n} |(-\Delta/4\pi^2)^{\rho/2} f|^2 dx \quad (21)$$

where $d\nu$ is surface measure (possibly weighted) on the surface S contained in \mathbb{R}^{mn} .

Practical application for the Klainerman-Machedon method and such convolution-type estimates has proved to be efficient by replacing the Riesz potentials with Bessel potentials on the Fourier transform side ([3], [5]); advantage is achieved by removing local singularities while gaining integrability on the potential side and improving the range of application as “smoothing operators”; still the lack of homogeneity limits determination of precise dependence on parameters in computing best size estimates. But as with exact model calculations, the role of Riesz potentials can result in “very elegant and useful formulae” that underline intrinsic geometric structure, capture essential features of symmetry and uncertainty, and provide insight to precise lower-order effects.

REFERENCES

- [1] W. Beckner, *Pitt's inequality with sharp convolution estimates*, Proc. Amer. Math. Soc. **136** (2008), 1871–1885.
- [2] W. Beckner, *Multilinear embedding estimates for the fractional Laplacian*, Math. Res. Lett. (in press).
- [3] T. Chen and N. Pavlovic, *On the Cauchy problem for focusing and defocusing Gross-Pitaevskii hierarchies*, Discr. Contin. Dyn. Syst. **27** (2010), 715–739.
- [4] C. Fefferman, *Inequalities for strongly singular convolution operators*, Acta Math. **124** (1970), 9–36.
- [5] K. Kirkpatrick, B. Schlein and G. Staffilani, *Derivation of the two-dimensional nonlinear Schrödinger equation from many body quantum mechanics*, Amer. J. Math. **133** (2011), 91–130.
- [6] S. Klainerman and M. Machedon, *Remark on Strichartz-type inequalities*, Internat. Math. Res. Notices, 1996, 201–220.
- [7] S. Klainerman and M. Machedon, *On the uniqueness of solutions to the Gross-Pitaevskii hierarchy*, Comm. Math. Phys. **279** (2008), 169–185.
- [8] N.S. Landkof, *Foundations of modern potential theory*, Springer-Verlag, 1972.
- [9] E.M. Stein, *Singular integrals and differentiability properties of functions*, Princeton University Press, 1970.
- [10] E.M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton University Press, 1993.
- [11] E.T. Whittaker and G.N. Watson, *A course of modern analysis*, Cambridge University Press, 1927.

DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF TEXAS AT AUSTIN, 1 UNIVERSITY STATION C1200,
 AUSTIN TX 78712-0257 USA
E-mail address: `beckner@math.utexas.edu`